PLASMA DISPERSION FUNCTION

Definition[16] (first form valid only for $\operatorname{Im}\zeta > 0$):

$$Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{+\infty} \frac{dt\, \exp\left(-t^2\right)}{t-\zeta} = 2i \exp\left(-\zeta^2\right) \int_{-\infty}^{i\zeta} dt\, \exp\left(-t^2\right).$$

Physically, $\zeta = x + iy$ is the ratio of wave phase velocity to thermal velocity.

Differential equation:

$$\frac{dZ}{d\zeta} = -2\left(1+\zeta Z\right),\ Z(0) = i\pi^{1/2}; \quad \frac{d^2 Z}{d\zeta^2} + 2\zeta \frac{dZ}{d\zeta} + 2Z = 0.$$

Real argument ($y = 0$):

$$Z(x) = \exp\left(-x^2\right)\left(i\pi^{1/2} - 2\int_0^x dt\, \exp\left(t^2\right)\right).$$

Imaginary argument ($x = 0$):

$$Z(iy) = i\pi^{1/2} \exp\left(y^2\right)\left[1 - \operatorname{erf}(y)\right].$$

Power series (small argument):

$$Z(\zeta) = i\pi^{1/2} \exp\left(-\zeta^2\right) - 2\zeta\left(1 - 2\zeta^2/3 + 4\zeta^4/15 - 8\zeta^6/105 + \cdots\right).$$

Asymptotic series, $|\zeta| \gg 1$ (Ref. 17):

$$Z(\zeta) = i\pi^{1/2}\sigma \exp\left(-\zeta^2\right) - \zeta^{-1}\left(1 + 1/2\zeta^2 + 3/4\zeta^4 + 15/8\zeta^6 + \cdots\right),$$

where

$$\sigma = \begin{cases} 0 & y > |x|^{-1} \\ 1 & |y| < |x|^{-1} \\ 2 & y < -|x|^{-1} \end{cases}$$

Symmetry properties (the asterisk denotes complex conjugation):

$$Z(\zeta^*) = -\left[Z(-\zeta)\right]^*;$$

$$Z(\zeta^*) = \left[Z(\zeta)\right]^* + 2i\pi^{1/2} \exp[-(\zeta^*)^2] \quad (y > 0).$$

Two-pole approximations[18] (good for ζ in upper half plane except when $y < \pi^{1/2} x^2 \exp(-x^2),\ x \gg 1$):

$$Z(\zeta) \approx \frac{0.50 + 0.81i}{a - \zeta} - \frac{0.50 - 0.81i}{a^* + \zeta}, \quad a = 0.51 - 0.81i;$$

$$Z'(\zeta) \approx \frac{0.50 + 0.96i}{(b-\zeta)^2} + \frac{0.50 - 0.96i}{(b^* + \zeta)^2}, \quad b = 0.48 - 0.91i.$$